DOCUMENT RESUME

ED 218 090 SE 037 884

AUTHOR Hunting, Robert P.
TITLE The Significance of Unit Knowledge for Children's Learning of Fractions.
PUB DATE Mar 82
NOTE 30p.; Paper presented at the Annual Meeting of the American Educational Research Association (New York, NY, March 18-23, 1982).

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS Basic Skills; Cognitive Development; *Cognitive Processes; Educational Research; Elementary Secondary Education; *Fractions; *Learning Theories; *Mathematics Education; *Number Concepts
IDENTIFIERS *Mathematics Education Research

ABSTRACT

Basic fraction concepts are viewed as the seedbed for many important mathematical ideas including the notions of equivalence, inverse, decimals, probability, ratio, and proportion. It is viewed that a critical component of children's mathematical knowledge necessary for success in certain fractional number contexts appears to be intellectual mechanisms for dealing with systems of units. Twenty-nine children were interviewed in May 1979 using sets of partition and fraction problems. This group consisted of 9 fourth-grade, 10 sixth-grade, and 10 eighth-grade children, all of whom were average or above average in mathematics. Data from seven of the fourth graders and two of the sixth graders were analyzed for this document. Among the results, it is found that the children who did not succeed at solving the given problems had not been able to take advantage of their knowledge of units, because either (1) such knowledge was not integrated into their existing fraction knowledge, or (2) knowledge of units was not the basis for whatever meanings these pupils had for fractional numbers. (MP)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED218090

THE SIGNIFICANCE OF UNIT KNOWLEDGE FOR CHILDREN'S LEARNING OF FRACTIONS

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY

Robert P. Hunting

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

U.S. DEPARTMENT OF EDUCATION
NATIONAL INSTITUTE OF EDUCATION
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

☑ This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

• Points of view or opinions stated in this document do not necessarily represent official NIE position or policy.

Robert P. Hunting

Western Australian Institute of Technology

Paper presented at the American Educational Research Association Annual Meeting, New York City, March 19-23, 1982.

SE037 884

Basic fraction concepts are the seedbed for many important mathematical ideas including the notions of equivalence, inverse, decimals, probability, ratio, and proportion. Cognitive structures which support fraction knowledge need to be understood because control over these numbers can allow the child to function effectively over an extended range of the quantitative environment.

A critical component of children's mathematical knowledge necessary for success in certain fractional number contexts appears to be intellectual mechanisms for dealing with systems of units. The notion of unit system, certain cognitive operational systems considered necessary for solving problems containing systems of units in discrete quantity partition contexts, and their potential importance for understanding how fractions are learned are discussed in the first part of this paper. The second part will present data contrasting the behavior of a small group of children in their attempts to solve certain discrete quantity partition problems and fraction problems.

One possible basis upon which fraction knowledge might be constructed is a set of actions resulting in "fair shares" of discrete quantities. The activity of constructing quantitative units or "shares" has received some attention in the literature as the basis for the development of a division algorithm (Clement, 1979; Irons, 1977). Kieren (1976) posited potential developmental and constructive mechanisms for the acquisition of rational number knowledge. Among these were partitioning, defined as any general strategy for dividing a given quantity into a given number of equal parts, and identification of the unit or whole. Kieren suggested that partitioning might play the same role in the development of a rational number construct as counting appeared to do in the case of the development of natural number knowledge.

Findings from several studies point to the possibility that knowledge of discrete quantity partitions including operativity with unit systems, is necessary for constructing knowledge about fractions and for solving fractional number problems. While Kieren did not distinguish partitioning strategies for sub-dividing discrete quantities from those for sub-dividing continuous quantities, this distinction would appear to

be important. There are grounds for rejecting the continuous case as a basis for developing fractional number knowledge in favour of partitions of discrete quantities. Although some research (Piaget, Inhelder and Szeminska, 1960) has determined a number of conditions needed for the effective production of units using continuous quantities such as clay, there is evidence (Hiebert and Tonnesson, 1977) suggesting that these conditions are not applicable to the subdivision of discontinuous quantities. Furthermore, findings reported by Schuerger and Burke (1977) and Mitchell and Blume (1977) suggest that knowledge of fractions based on actions of subdividing continuous quantities is inadequate. Children faced with collections of discrete objects did not know how to construct appropriate representations for simple unit fractions.

Theoretical Framework: The Construct of Unit System

The possibility that certain operational structures needed for dealing effectively with systems of units might be important for children who are learning and applying fraction knowledge was suggested as a result of a theoretical analysis of the role of unit systems in the context of certain fraction problems (Hunting 1981a). A unit system is the set of relationships which exists between two or more unit types. Seven types of units that are relevant to this research may be distinguished (see Figure 1):

1. Units as singletons. These are "objects" which exist for children in perception as discrete (whole) entities, or in figural, internal representation, as discrete (whole) entities. These are units which support the construction of whole number knowledge.

2. Units as definite numbers of singletons. These are "objects" which exist for children as a result of taking a particular number of singletons as a unit.

3. Units from subdividing a definite number of singletons. These are "objects" which exist for children as a result of subdividing a definite number of singletons.

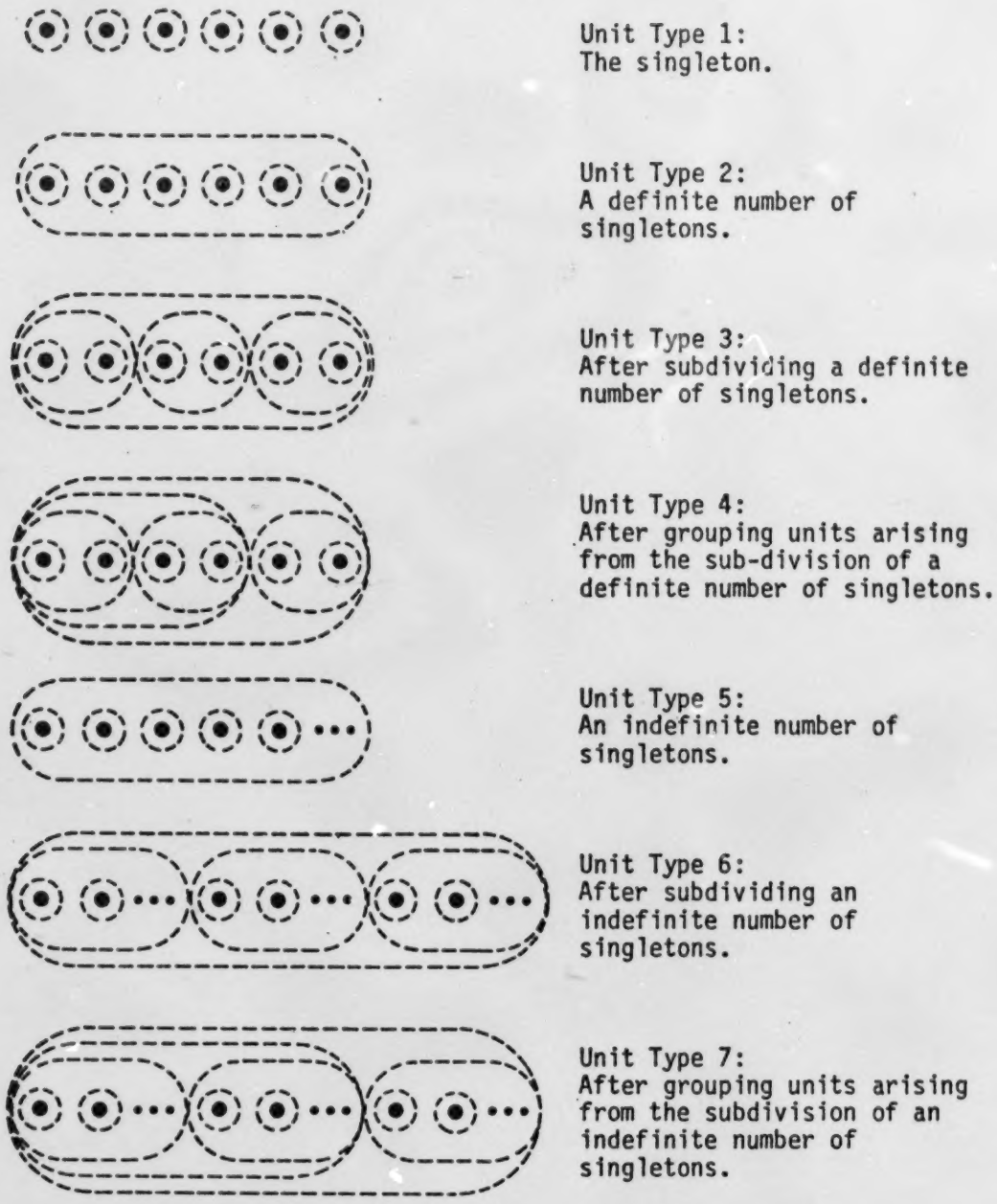

Figure 1. Types of Units

4. Units as multiples of units arising from the subdivision of a definite number of singletons. These are "objects" which exist for children as a result of grouping together a number of units arising from subdivision of a definite number of singletons.

5. Units as an indefinite number of singletons. These are "objects" which exist for children as a result of taking an indefinite number of singletons as a unit.

6. Units from subdivisions of an indefinite number of singletons. These are "objects" which exist for children as a result of subdividing an indefinite number of singletons.

7. Units as multiples of units arising from subdivisions of an indefinite number of singletons. These are "objects" which exist for children as a result of grouping together a number of units arising from a subdivision of an indefinite number of singletons.

- - - - - - - - - - - - - - - -

Insert FIGURE 1 about here

- - - - - - - - - - - - - - - -

Two unit systems can be identified from the list of unit types just described. Unit Types 1, 2, 3 and 4 can together form a system of relationships in which each unit type is known in terms of a definite number of singletons. Unit Types 1, 5, 6, and 7 can relate to each other in a system, but in this case the number of singletons in Types 5, 6 and 7 is unknown. These two unit systems differ from one another with respect to a definite-indefinite dimension which characterizes units consisting of pluralities of singletons.

As an exemplification of this distinction, consider the following two problems. First, a child has the task of distributing a given number of candies equitably among four friends. In this context three types of units can be distinguished. The individual candy is a singleton or or Type 1 unit. The total number of candies to be shared is an example of a Type 2 unit. It is a unit consisting of a definite number of

singletons. The solution to the problem is found by determining the number of candies each of the four friends will receive. The number of candies in each share is an example of a Type 3 unit. Such units arise from the subdivision of a definite number of singletons.

Second, suppose that a child and an adult are each given a box containing an equivalent but unknown number of discrete objects. This problem is based upon a supposition that the contents of the child's box can be stacked into a number of equal piles on the table. Let us call this number of piles x. The contents of the adult's box can be also stacked up, but into a different number of equal piles - say, y piles. The child is required to find the number of the adult's y equal piles which would be a fair trade for, say, three of the x equal piles. In the context of this problem the Type 1 unit, the discrete object, is not available but exists implicitly. The two boxes which contain equivalent quantities of objects are Type 5 units. These are units containing indefinite numbers of singletons. The hypothetical piles of objects are examples of Type 6 units, since the contents of each box has been subdivided. In this problem there are two different Type 6 units. The units arising from subdividing the contents of one box into x equal piles are not the same size as the units arising from the subdivision of the contents of the other box. This knowledge is critical for the solution of the problem. The Type 7 unit in this problem is represented by three of the x equal piles, or the required number of the adult's y equal piles.

In order to describe some cognitive operational systems that may be sufficient to allow a child to successfully solve problems involving either or both of these unit systems, a mathematical analysis of structural relationships inherent in partitions of collections of objects is provided.

Supporting Mathematical Theory

Mathematically, a partition P of a set S is a set of non-empty subsets of S that is disjoint and exhaustive. That is, for $S \neq \phi$, P is a subset of the power set of S such that:

(a) if $X \in P$, then $X \neq \phi$;

(b) if $X_1, X_2 \in P$, with $X_1 \neq X_2$, then $X_1 \cap X_2 = \phi$; and
(c) $\bigcup_\alpha X_\alpha = S$.

Of particular interest in this study are those partitions P of a set S where the $X_\alpha \in P_\alpha$ are restricted by the condition $|X_i| = |X_j|$, $0 < i, j \leq \alpha$, $i \neq j$. That is, the elements X_α of the partition P_α, which are subsets of S, are equivalent.

Certain properties of such partitions follow. Note that $\alpha = |P_\alpha|$ will be used to denote the number of subsets of the partition P_α, and $|X_\alpha|$ to denote the magnitude of each of the subsets $X_\alpha \in P_\alpha$.

For a set S, with $|S|$ finite:

(a) $|X_\alpha| \cdot \alpha = |S|$;
(b) $P_\alpha = \{X_\alpha : X_\alpha \subset S$ exists if and only if α divides $|S|$.

It follows that for each distinct divisor of $|S|$ there is a partition;

(c) For $P_\alpha = \{X_\alpha : X_\alpha \subset S$ and $P_\beta = X_\beta : X_\beta \subset S\}$, with $\alpha \neq \beta$, since $|X_\alpha| \alpha = |S| = |X_\beta| \beta$, it follows that $\alpha/\beta = |X_\beta| / |X_\alpha|$.

Consequently,

(d) $\beta < \alpha$ if and only if $|X_\alpha| < |X_\beta|$; and
(e) β divides α if and only if $|X_\alpha|$ divides $|X_\beta|$; the implied quotients being identical.

Property (a) implies that the number of subsets, α , in the partition P_α of a set S can be found by dividing the magnitude of each subset, $|X_\alpha|$, into the number of elements in the set, $|S|$, and similarly, $|X_\alpha|$, the magnitude of the subset, can be found by dividing α, the number of subsets, into the number of elements in the set, $|S|$. Property (b) implies that a non-empty set S can be partitioned at least two ways, depending on the number of divisors of $|S|$, the number of elements in the set. Property (c) shows that a proportional relation exists between the ratio of the numbers of subsets, α and β , and the sizes of the subsets, $|X_\alpha|$ and $|X_\beta|$, of partitions P_α and P_β on a set S. It is an inverse proportion. Properties (d) and (e) follow from property (c). Property (d) shows the qualitative relationship which relates the sizes of the

subsets between two partitions given the numbers of these subsets. Property (e) shows that the relationship between the sizes of the subsets can be quantified given the numbers of the subsets in each partition, and vice versa.

The supporting mathematical theory will now be drawn upon to suggest hypothetical models of operational systems which might be manifested in the reasoning patterns of children dealing with unit systems in the context of certain discrete quantity partition situations.

Operational Systems: Definite Number in the Unit

Consider a problem where four piles, consisting of six cookies each, are placed in front of a child. The child is required to determine the size of the piles which would be needed if just 12 dolls were to share the cookies evenly. Property (a) from the mathematical theory implies the following solution: Four piles of six cookies lie on the table. Therefore, by multiplication, the total number of cookies in the partition is 24, since $|X_\alpha|\alpha = |S|$; where $|X_\alpha|$, the unit size, is known to be six cookies, and , the number of units is four. To find the magnitude of $|X_\beta|$ in the new partition P_β, property (a) can be used again. The number of new units β, which is given to be 12, can be divided into $|S| = 24$ to obtain $|X_\beta|$, the size of the new pile.

Property (b) states simply that since the number of the dolls, which is 12, divides the total number of cookies, a solution to the problem exists. That is, each of the 12 dolls is guaranteed to receive an equal share. On its own, property (b) does not appear to contribute to a solution to this problem.

The proportional relation found in property (c) suggests that the problem could be stated in this way: Four piles of six cookies is equal to 12 piles of how many cookies. The values of α, β and $|X_\alpha|$ in the relationship $\alpha|X_\alpha| = \beta|X_\beta|$ are known. The value of $|X_\beta|$ is not. Property (d) indicates that a form of compensation reasoning (Hunting, 1981b) might be used to indicate the qualitative magnitude of the resultant piles compared with the initial piles. That is, an increase in the number of piles is compensated for by a decrease in the size of

each pile. Therefore, since the number of piles in this problem becomes greater than 4, the size of each resultant pile must become smaller than before; that is, less than 6 cookies. Property (e) contributes precision to the preceding argument. Since the final number of piles, 12, becomes three times greater than the initial number, 4, the magnitude of each resultant pile must become three times smaller. The amount of increase or decrease in number of piles, or correspondingly, the amount of decrease or increase in the size of each pile, can be quantified using the operation of division. This quotient is the proportionality constant. Knowing the number of cookies in each beginning pile means that the proportionality constant can be used to determine the number of cookies in each resultant pile directly, without the intermediate step of finding the total number of cookies involved in the problem.

Either of two operational systems then, may be used to direct the reasoning processes of a child in solving a partition problem like that described above. The more abstract reasoning pattern is governed by knowledge of the compensating relation between the variables of unit size and the number of units, either across a partition transformation, or between distinct partitions on equivalent sets of objects. Properties (c), (d), and (e) of the mathematical theory describe this relation. Property (c) suggests that a proportionality structure underpins relationships between sizes and numbers of units in a partition context.

An alternative operational system which might be used to direct solutions of partition problems involving real and material objects is indicated from property (a) of the mathematical theory. The relationship $|X_\alpha| \alpha = |S|$ for a partition P_α on a set S can be used here. In this case all units can be quantitatively related because the magnitudes of such units are perceived as definite numbers of singletons, or Type 1 units. The relationships between units consisting of definite numbers of singletons can then be computed precisely using whole number operations such as counting, multiplication, and division. The numerical values attached to particular defined aggregate units would refer to the number of Type 1 units contained therein. Of particular importance for this system is the total number of singleton units, $|S|$, present in the problem.

Operational Systems: Indefinite Number in the Unit

Consider a problem in which an adult and a child each have a box containing an unknown, but finite number of cookies. The child is given information that each box contains the same number of cookies, and is allowed to inspect but not count, the contents of each box. The child is then asked to imagine the contents of each box stacked into different numbers of piles of cookies. The adult's box is to be made into 8 equal piles; the child's box is to be made into 12 equal piles. Suitably marked cards are provided to indicate the appropriate number of piles. The child is required to find how many of the adult's 8 piles would be a fair trade for 3 of the 12 piles. In this setting, partitions are constructed on each of two equivalent quantities of objects. There are 8 units in one partition, and 12 units in the other.

An examination of the mathematical properties of partitions indicates that properties (a) and (b) cannot contribute to the solution of the problem because $|S|$, the total number of cookies originally contained in each box, is not known. Therefore, the exact number of cookies in each of the 12 piles cannot be calculated using whole number knowledge. This in turn prevents a quantification of the 3 piles to be traded in terms of a number of single cookies.

Properties (c), (d), and (e), which describe the structural relationships between the numbers of units, and the sizes of units between two partitions, make it possible for a solution to this problem to be found. Property (d) indicates that since the adult's eight piles are less in number than the child's 12 piles, each of the 8 piles must be larger than each of the child's 12 piles. Therefore trading for 3 of the 12 smaller piles can only be fair if less than 3 of the 8 larger piles is chosen, because 3 of the 8 equal piles would consist of more cookies than 3 of the 12. The compensating relation between the numbers of units and the sizes of the units in partitions on two equivalent quantities can be applied here at a qualitative level. In the context of this particular problem the answer could then be guessed or estimated from the choices available; one or two of the 8 piles.

Property (e) indicates that the ratio of the numbers of piles is inversely proportional to the ratio of the sizes of the piles. The constant of proportionality in this problem is two-thirds. That is, the number of the adult's piles is two thirds the number of the child's piles. So the size of the adult's piles must be one and one-half times the size of the child's. Quantification of the relative sizes of the piles is crucial in the process of solving the problem. Once the size of the unit in one partition is known in terms of the size of the unit in the other partition, there is a basis for equating quantities of cookies structured as different sized units. For example, in this problem, since the adult's piles are one and one-half times larger than the child's, it might be reasoned that if one of the 8 piles is equivalent to one and one-half of the 12, then two of the 8 is equivalent to three of the 12, -- the number of the child's piles to be traded.

Operational Systems and Fractions

In task situations requiring the construction of physical representations of fractions, functionality with a system of units seems necessary. Also, task situations involving problems of fraction equivalence may require knowledge necessary to relate with precision different units within a unit system.

A suitable framework in which to consider children's behavior with different types of units and their relations was chosen to be discrete quantity partitions. A careful study of children's behavior in particular partition contexts was planned to expose and clarify mental functioning needed to deal effectively with different systems of units. Initially it was necessary to identify and classify strategies children exhibited in solving discrete quantity partition problems where the units consisted of definite numbers of singletons, and strategies children exhibited in solving discrete quantity partition problems where the units consisted of definite numbers of singletons, and strategies children exhibited in solving partition problems where the units consisted of indefinite numbers of singletons.

The classification of strategies for dealing with partition problems involving systems of units provided a basis for understanding the nature of mental mechanisms responsible for particular strategies. To investigate the claim that mental operations for dealing with systems of units can provide a basis for the child to exert effective control over fractional number events, a set of fractions problems was devised. These problems were designed to allow children to demonstrate solution strategies in a variety of contexts including the construction of physical representations using discrete objects, the production and verification of equivalence expressions, and problematic situations involving fraction equivalence. The kinds of strategies and justification arguments offered in support of solutions were examined for evidence that would support or discount the claim that knowledge of fractions based on units or multiples of units and their relations is necessary for solving certain fractional number problems.

Procedures

Two interviews were constructed containing sets of discrete quantity partition problems and fraction problems. The problems were grouped according to task structure, and in general increased in complexity. For example, in the partition interview, problems involving discrete objects which were available to the child preceded problems where the objects were not available. The fraction interview began with problems of representing a single fractional quantity before introducing equivalence problems with two fractions. A summary of all problems is given below. However, only results from a selected set of problems are reported and discussed in detail in this paper.

Overview of the Partition Problems

Task P1: A child is shown some dolls and given a quantity of cookies. The number of cookies supplied is also given. The child is directed to give each doll the same number of cookies. The problems range from sharing 12 cookies between four dolls to sharing 51 cookies between three dolls.

Task P2: A quantity of cookies is provided together with an exemplary share for one doll. The problem is to determine how many

dolls can share the given quantity if each doll's share is pre-determined.

Task P3: Twenty cookies are placed on the table. The child is asked first to determine a number of dolls that could share the cookies evenly, and then to find as many other solutions as possible.

Task P4: Eight piles of three cookies are to be transformed into shares for four dolls. The child is asked first to describe how this might be done, then if necessary to construct the four piles. A problem where four piles are to be shared between 12 dolls follows.

Task P5: Each of five equal piles of freshly baked cookies is subdivided into three smaller piles to facilitate cooling. Although objects representing cookies are not actually present on the table, display cards showing marked places to represent the initial number of piles are shown. If six of the 15 new piles are placed in the cookie jar, how many of the old piles would that be? A similar problem follows where 12 equal piles of cookies are stacked into four piles to make more space on the table.

Task P6: Hypothetical piles of crackers are to be transformed into a different number of equal piles. The number of crackers per pile can not be calculated, but marked cards are available which show places corresponding to each number of piles. The problem are: 4 piles to 2 piles; 9 piles to 3 piles; 3 piles to 12 piles; and 6 piles to 4 piles. The child is asked to determine how much larger or smaller the transformed piles will become.

Task P7: The investigator and the child are each supposed to have the same number of cookies in identical containers. The investigator's cookies are made into six (imaginary) equal piles; the child's cookies are made into three (imaginary) equal piles. Marked cards show places corresponding to piles for each participant. Problems of trading piles are posed. For example: How many of the investigator's six piles would be a fair trade for two of the child's three piles? Trading problems involving eight and twelve equal piles follow.

Overview of the Fraction Problems

Tasks F1-F4: On the table are displayed a number of cardboard cutouts representing Easter eggs. These are all the eggs hidden prior to an egg hunt. The child is asked to imagine having found a fraction of the set of eggs shown, and to find the number of eggs representing

that fraction. The fractions used are 1/2, 1/3, 2/3, and 3/5. Supplementary problems were prepared for each task. For example, in Task F2, if the child could not find one-third of 12 green eggs, the investigator would repeat the problem using three yellow eggs, then six blue eggs. If the child succeeded with these simpler problems the original question was then repeated.

Tasks F5-F8: These problems are also set in the context of an Easter egg hunt. Here the child is given a number of colored eggs and told they are a certain fraction of all the eggs of that color that are hidden. For example, in Task F8, the child is shown six red eggs and told that these are three-fifths of all the red eggs. The problem is to find how many red eggs were hidden all together. As in Tasks F1-F4 the fractions chosen were 1/2, 1/3, 2/3, and 3/5, and supplementary problems were used when appropriate.

Task F9: The child is shown a fraction written on a sheet of paper, and asked to complete an equivalent fraction whose denominator is given. Then, using a supply of counters, the child is asked to verify that the two fractions are equivalent. Problems given included 1/2 = /4; 1/4 = /8; 1/4 = /12; 2/6 = /3, 6/15 = /5, and 3/12 = /8.

Task 10: The child is asked to represent a fraction written on a flashcard using a given number of counters. Then the child is asked to name another fraction compatible with the representation just constructed. If the child is able to respond, the investigator proceeds to ask why the two fractions are equivalent. If the child can not find an equivalent fraction in the representation, the investigator supplies one, and again asks why they are equivalent.

Tasks F11, F12: These problems were assumed to require fundamental equivalence knowledge for successful solutions. In the first task, two sets of M & M packets are shown; each set containing a different unit fraction of their original, equal, contents. The first set contains one-fourth of the original contents; the second set contains one-eighth of the original contents. The problem involves the child trading four of the one-eighth full packets for some of the one-fourth full packets. A follow-up problem involves M & M packets containing one-half and one-sixth of their original contents. The context for Task F12 is two coins in a foreign currency which are given to be different fractions of a single dollar. The fractions are one-fifth and one-tenth. In place

of the numerals on the coins are signs showing the given fractions. The problem posed involves the child in trading a given number of one type of coin for some of the other type.

The interview problems were conceived of as power instruments; therefore it was expected that not all children would succeed on all aspects of each set of problems. Contingency plans were specified to ensure that children were not administered problems which were obviously beyond their capacity to solve successfully. However, overriding decisions were made by the investigator in the light of the situation as it was seen at the time of interview.

Both sets of problems were administered individually in the school environment, but in a private area. The duration of each interview sitting varied from child to child, but averaged 35-40 minutes. In some cases it was necessary to complete a set of problems over two sittings. The investigator exercised flexibility in the interview situation, within the limitations of the interview plans, in response to the abilities and personality characteristics demonstrated by the child, and the local influences existing in the school from day to day. The interview plans were tightly scripted to ensure a standard presentation of each problem, and to indicate follow-up questions to anticipated responses. Each interview was recorded on audio-video tape.

The Sample

Twenty nine children were interviewed in May, 1979, using the sets of partition and fraction problems. This group consisted of 9 fourth grade, 10 sixth grade, and 10 eighth grade children, all of whom were of average or above average ability in mathematics. Data obtained from 7 of the fourth grade and 2 of the sixth grade children were analyzed for this report.

The fourth grade children were drawn from a single composite fourth and fifth grade class at an elementary school in Clarke County, Georgia. Their ages ranged from 9 years 10 months to 10 years 5 months. Children of this age were selected because it was expected that their behavior would provide valuable insights into the genesis of rational number

knowledge, and the role that partitioning might play in the construction of such knowledge. A minimum of fraction instruction had been received. No formal lessons had been given during the current school year up to the time of interview. After a preliminary examination of the fourth grade data, seven of the nine children were selected using the criterion of maximizing the variation of overall competency on each set of problems. These children were found to have very little knowledge of equivalence of fractions. Consequently, two sixth grade children from another elementary school in Walton County, Georgia, were chosen. These children, aged 12 years 1 month and 12 years 3 months, demonstrated a relatively higher degree of competence in this area.

The Analysis

Procedures for analyzing the data were developed through two phases. These were (a) the development of a set of working transcripts from the audio-video tapes; (b) a task by task analysis of the behaviors of the children taken as a group for both the partition problems and the fraction problems.

In the development of a set of working transcripts for each of the nine children two kinds of data were translated. Verbal transactions between the child and the investigator were transcribed. In addition, non-verbal communications occurring concurrently with the flow of conversation were described. These included action sequences involving manipulation of objects present in the task setting, as well as gestures and signs expressed by the child in the course of executing or justifying a problem solution. Whereas the transcribed exchanges tended to be a literal report of the tape record, non-verbal descriptions were necessarily less literal. Judgments were made about which activity to report and which activity to ignore. The criterion for selecting non-verbal activity for inclusion in the working transcripts was that the flow of verbal transactions be made as unambiguous as possible.

The purpose of the analysis by task was to identify and classify solution strategies exhibited for each task type, and to elaborate mental mechanisms responsible for observed behaviors. An advantage of this approach was that for a particular problem, a child's behavior could be

interpreted with reference to the catalog of behaviors displayed by the other children. The analysis by task was prepared using working transcripts, and forms the basis of the results reported here.

Results

Two analyses follow. Each analysis will contrast the solution strategies offered by a group of nine children in their attempts to solve two tasks having structural similarities. In the first analysis the tasks involve units containing definite numbers of singletons. One task was given in the context of a discrete quantity partition interview (Hunting, 1981a). The other task was given in the context of a fraction interview. In the second analysis the tasks involve units containing indefinite numbers of singletons. Again, the context for one task is a discrete quantity partition interview, and the context for the other is a fraction interview.

Contrast I: Units containing definite numbers of singletons

Task P1: Constructing a partition. This task contained problems of sharing cookies between dolls (see overview of the partition problems). Three types of unit are present in the problems in this task. The individual cookie corresponds to the singleton, or Type 1 unit (see Figure 1). The total collection of cookies to be shared is a unit consisting of a definite number of singletons. As such, this represents a Type 2 unit. The shares of cookies, which each of the dolls is to receive are Type 3 units. These units arise out of the subdivision of a definite number of cookies. These problems, then, contain three unit types. The three unit types are related to one another numerically and so form a unit system.

Two kinds of strategies observed or inferred from the behavior of the children studied in Task P1 were (a) anticipatory solutions based on whole number knowledge, and (b) physical experiments where the units were constructed directly using the available counters. Anticipatory strategies included the following:

1. Immediate recall of the appropriate number fact. This method is inferred from the actions of unhesitatingly counting out the correct number of cookies for the first doll, then counting out the cookies for the next doll, etc., until each doll has the correct number of cookies. All of the children studied used number fact knowledge on at least one occasion.
2. Construction of needed whole number knowledge. A variety of strategies was used here, exhibited mostly for the problem of sharing 51 cookies between three dolls. The most frequently used method was the long division algorithm. Six children used it. One other knew it was appropriate but failed to execute it. Other strategies used included (a) estimation and repeated subtraction involving partial quotients,(b) an estimation procedure whereby a number selected as a candidate for the unit size was written three times in a column and summed, and (c) a method of checking successive multiples of three mentally.

Physical experiments where the units were constructed directly using the available counters were of the following types:

1. Systematic equalization with conservative estimation of unit size. Here the child selects a number of cookies which can be equalized across all the dolls so that the supply of cookies is not exceeded. A further estimate is then made from the remaining cookies. This method has the advantage that a running tally of cookies allocated can be kept easily by keeping track of the sizes of the units incremented.
2. Systematic equalization with single cookie increments. With this method a single cookie is allocated to each doll in turn on a cyclic basis.
3. Equalization using trial and error. Here the child constructs a unit in front of the first doll. A judgment based on perceptual information is made concerning how large the unit should be. This unit then becomes the exemplar and an attempt is made to construct similar units in front of the other dolls. An overestimate could cause the child to run out of cookies before completing the last pile.

Two different kinds of strategies were used to solve these tasks. The first kind is distinguished by anticipatory procedures based on whole number knowledge. These procedures are directed toward pre-determining the magnitude of the unit in a partition represented by the share of cookies each doll might receive, or the number of units in a partition represented in these tasks by the number of dolls which might receive a particular number of cookies from a given supply. The second kind is distinguished by physical experiment on the part of the child as the unit sizes and numbers of units are constructed using real and material objects.

The significance of the distinction between anticipatory whole number operations and the constructive physical experiment is discussed in Hunting (1982).

Task F2. Constructing Part from Whole: 1/3. This problem required the child to find one-third of a set of 12 objects representing Easter eggs (see overview of the fraction problems). Immediately prior to this problem each child had successfully solved a similar problem of finding one-half of 10 Easter eggs. Success in this kind of task was assumed to involve the coordination of at least three types of unit. The individual Easter egg represented the singleton, or Type 1 unit (see Figure 1). The total collection of eggs was a definite number of singletons, or Type 2 unit. The problem was to create Type 3 units from the Type 2 unit. The nature of the Type 3 unit could be determined using knowledge about the given fraction.

Five of the seven fourth grade children studied did not know the fraction 1/3 based on their spontaneous responses. Here is the evidence:

AD (9;11) could not solve the problem of finding one-third of 12 eggs, so the investigator repeated the question using three eggs. AD chose one egg. But for the same question using six eggs, AD selected three eggs to represent the fraction 1/3.

EF (10;2) knew one-half of 10 eggs was five eggs. He argued that five plus five was equal to 10. But he admitted not knowing the meaning of one-third. EF was assisted to find one-third of 12 eggs by first being told one-third meant one of three equal parts or groups and, when

this did not seem to help, by being given explicit directions to make the 12 eggs into three equal groups. This he did using a systematic one to one equalization strategy. When the investigator placed nine eggs in a container, EF said that one-third of these would be four because one plus three was four.

LL (9;11)'s idea of one-third was tied to cutting a pie into fourths, and removing three parts. The three parts represented one-third. The investigator corrected her, and reinterpreted one-third as one of three equal pieces of pie. LL could not transfer this information to the context of finding one-third of three eggs. She said two eggs. After the investigator corrected this error, LL said one-third of six eggs was three eggs. When asked what one-third meant, LL returned to her pie interpretation.

MC (10;5) said five eggs was one-half of 10 eggs because five plus five equaled 10. She did not know the fraction 1/3. In Task F2 she did not respond to the question of finding one-third of 12 eggs. She identified one-third of three eggs correctly, but said three eggs represented one-third of six eggs. The investigator then helped MC find one-third of six eggs. MC was then able to find one-third of 12 eggs unassisted.

SR (10;0) solved Task F1 immediately, giving the division fact $10 \div 2 = 5$ as justification. SR's conception of the fraction 1/3 was found to be three of four equal slices of pie. The investigator assisted her with a meaning for one-third as one of three equal parts. SR interpreted this in a literal sense. She found four eggs to be one-third of 12 eggs by separating the 12 eggs into groups of three, and removing one egg from each group.

Contrast II: Units containing indefinite numbers of singletons

Task P6: Relations involving Quantitative Compensation. The problems in this task involved the transformation of a known number of hypothetical piles of crackers to a different number of piles (see overview of the partition problems and Figure 2). The system of units contained in each of these problems consisted of Type 1, Type 5, and two different Type 6 units (see Figure 1). The Type 1 units, represented by

single crackers, existed implicitly in the problem since they were never actually available. The contents of the cracker box, which consisted of an indefinite number of crackers, was a Type 5 unit. Arising from the subdivision of the contents of the cracker box were Type 6 units. These were the piles of crackers. The contents of the box were initially partitioned into a given number of equal piles of crackers, from which a different number of equal piles of crackers was constructed.

These problems involved the child in using the numbers of Type 6 units in the two partitions to relate the unit sizes. Qualitative compensation (Hunting, 1981b) seemed necessary to answer the first question in the problem, in which the child was asked what would happen to the piles as a result of the transformation. The second question, in

---

Insert FIGURE 2 about here

---

which the child was asked to explain how much larger (or smaller) the transformed piles had become compared with the original piles, seemed to require a quantification of the relation between the pile sizes. Prior to this task, the investigator had worked through a number of warm-up activities with the children designed to review appropriate language for describing the differences in magnitudes of piles of counters.

Two types of mental functioning suggested by the strategies exhibited by the children across the four problems in this task were:

1. Limited quantitative compensatory comparisons. In this category were children who were able to quantify the increase or decrease in the sizes of the transformed piles using the numbers of piles before and after the transformation. Appropriate solutions were limited to whole number transformation factors. Each of the children in this category offered quantitative solutions consistently across problems 6A, 6B, and 6C. But these solutions appeared to break down at problem 6D (see Figure 2) which involved a non-integer transformation factor. For example, MC (10;5) knew that the piles would not become twice as large because she realized that either six piles would have to become three, or eight piles would have to become four. She attempted a special case solution with

Figure 2: "And suppose that you are able to take all the crackers in the six piles and make them into just four equal piles (Problem 6D)."

two crackers per pile but finally had to carry out the transformation with four actual crackers on each of the six piles.

2. Qualitative compensatory comparisons. Children in this category were able to say that the piles would become larger or smaller on the basis of a comparison of the number of piles before and after the transformation, but failed to consistently quantify the relation between the initial and transformed piles. Five children fell into this category, based on the following evidence:

AD (9;11) gave quantitative solutions to problems 6A and 6B supported by explanations of how the piles would be stacked. She first guessed six times shorter for problem 6C. When asked if there was a way to be sure, AD invoked the division fact 12 ÷ 3 = 4 to say that the piles would become four times shorter. In problem 6D the piles were said to become twice as large. AD justified this conclusion in terms of the difference between six and four.

CP (10;3) explained the outcome of problem 6A, where four piles become two piles by pretending to combine pairs of imaginary piles. In problem 6B, where nine become three piles, he said the piles would become twice as large because two piles would be stacked on top of a third to carry out the transformation. He answered four times smaller in problem 6C. In his argument he gave the multiplication fact 4 x 3 = 12 and explained that the row of three piles would have to be duplicated four times to make 12 piles. In problem 6D, where six piles became four piles, CP acted out the transferral of four out of the six imaginery piles across to the four place display. But he concluded that the problem wasn't solvable because the two remaining piles, when transferred, would make the four piles unequal.

EF (10;2) solved problem 6A for the special case where each of the four piles contained just two crackers. For problem 6B, EF indicated that the three equal piles would be larger in size. He showed with his hands how much larger he expected these piles to become, together with the verbal estimate 27 times larger. He was not given problem 6C and 6D.

KB (12;3) said the piles would become four times larger in problem 6A because "four piles go into two piles". In problem 6B the piles were said to become three times larger. KB gave a ratio argument, saying that it would take three of the nine piles to make one of the three. In

problem 6C she said the piles would become 12 times smaller. KB justified her answer of two times larger in problem 6D by acting out the transformation with imaginary piles and parts of piles. While she spoke of splitting piles up, KB did not seem to realize that half piles would be created.

LL (9;11) gave quantitative answers to problems 6A and 6B. In problem 6C LL first said the piles would be twice as small, then three times smaller. She focused on the number of marked places in a row on the 12 place display, which was three. In problem 6D the piles were said to become twice as large, since six divided into four was two, according to LL.

One child, DD (9;10), was found to be a transitional case between the two categories defined above. He offered quantitative solutions in problems 6A and 6B, although his subsequent behavior in problem 6C indicates that his solution strategy for the first two problems was based on a visual inspection of the columns formed by the marked places by holding the rows constant across the display cards. However, in problem 6C he was forced to abandon this strategy, and concluded that the piles would become four times smaller, using the knowledge that the 12 equal piles would be smaller in size than the three, in conjunction with the multiplication fact 3 x 4 = 12. In problem 6D, DD chose to act out the transformation using half piles as the unit. He rejected the suggestion that the piles would become twice as large, because he knew that the quantity would be a fraction. He concluded that the piles would become a half larger.

Task F11, F12: Problems involving equivalence. In Task F11 the children were asked to find how many one-fourth full M&M packets would be a fair trade for four one-eighth full M&M packets (see overview of the fraction problems). Task F12 was similar, but involved trading coins in a foreign currency designated to be one-fifth and one-tenth of a dollar.

The system of units contained in Task F11 consisted of Type 1, Type 5, Type 6 and Type 7 units (see Figure 1). The Type 1 units were the individual M&M's, which were not available to the child. Type 5 units were the original contents of each packet of M&M's--an indefinite number

of singletons. Type 6 units were the fractional parts of each M&M's packet given as unit fractions 1/4 and 1/8, and 1/? and 1/6. A Type 7 unit was the quantity of M&M's contained in four of the one-eighth full packets to be traded. Arguments employed in comparing one fractional quantity with another, especially the supporting knowledge allowing the relationships between the fractions to be quantified, were of particular interest.

Appropriate strategies that were exhibited by the children in their solution attempts were found to be:

1. Quantitative comparisons of unit fractions. MK (12;1), in justifying his answers to each of the problems, stated accurately the size relationship between the fraction pairs. For example, in Task F11, he correctly answered that six of the one-sixth full M&M's packets would contain as much as two one-half packets because, "the amount in this (indicating the 1/2 flashcard) would be three times as much as this (indicating the 1/6 flashcard) so you'd have to get three times as many of those".

2. Comparisons based on the inverse relation between the size of the unit fraction and the cardinality of the denominator. LL (9;11) in Task F11 said that the one-fourth full packets contained more M&M's than the one-eighth full packets. Her justification, "because one-fourth, one eighth is smaller than one-fourth, but it sounds like one-eighth is bigger than one-fourth but it's really not", shows that she was aware of the contradictory nature of the actual size relation between each unit on one hand, and the figural information given by the denominators, on the other. In Task F12 LL chose the fraction 1/5 as being larger than 1/10 because the 10 was greater than five.

3. Application of partition knowledge. In Task F11 KB (12;3) solved the follow-up problem reasoning that one-half and one-half made up a whole, and that the number of one-sixths to make up a whole would have to be six. She argued in Task F12 that the one-fifth dollar coin would have greater value by explaining that the units associated with the fraction 1/10 would have to be smaller, when compared with the one-fifth units.

4. Use of ratio knowledge. In Task F11 CP (10;3) was observed to recall the solution to a previous problem 1/4 = ?/8 from Task F9 and

repeat the argument that since four was added to the denominator of the fraction 1/4 to make eight, one needed to be added to that fraction's numerator to give the missing number.

5. Other strategies. KB (12;3) in Task F11 wrote on paper the expression 2/4 = 7/8, and solved it correctly. It is not known if she used cross-multiplication or common factorization as she was observed to do in Task F9. SR (10;0) was given assistance with the follow-up problem in Task F11. She was told that each packet contained six M&M's when full. SR used arithmetic to calculate how many one sixth full packets would be needed.

Inappropriate strategies exhibited in the context of these tasks were the result of interpreting the denominator of a fraction as its magnitude. Thus, in Task F12 LL (9;11) indicated that the fraction 1/5 was larger than the fraction 1/10 because five was smaller than 10. Five other children offered solutions on the basis of figural information contained in the denominators. In Task F11, AD (9;11), DD (9;10), and MC (10;5) indicated that they believed the one-eighth and one-fourth full M&M's packets to contain eight and four M&M's, respectively. EF (10;2) gave no reason why he thought the fraction 1/4 was smaller than the fraction 1/8. CP (10;3) said that four one-sixth full packets would contain the same number of M&M's as two one half full packets because the difference between the denominators of the two fractions was four.

The behaviour of MK (12;1) in quantitatively comparing the fractions 1/2 and 1/6 is evidence that he was capable of engaging mental operations necessary for the coordination of units in a system, where the units consisted of indefinite numbers of singletons. It is reasonable to suppose that one-half and one-sixth were, for MK different kinds of units. MK had a basis for fraction knowledge founded in partitioning. His behavior in the context of Tasks F1-F8 showed this. Although it is not known <u>how</u> MK came to conclude that one-half was three times larger than one sixth, such a solution suggests the existence of an operational system for dealing with units centered on compensation knowledge.

KB (12;3) gave a compensation argument in Task F12 for the fraction 1/5 being larger than one-tenth. The following transcript provides

strong evidence that she was able to extend qualitative compensatory logic a step further to quantify the relationship. The investigator asked which of the two coins would buy the most.

KB: "This one (one-tenth coin)."

Investigator: "How do you know that?"

KB: "No. It's got to be this one (indicating the one-fifth coin) because one-fifth, if you had 10.... one-tenth would have to have smaller because there'd be more unit, I'll call them units, and one-fifth would buy more, because there'd be less units and if you compared those together one-fifth you'd buy more."

Investigator: "Well, how many of these coins (one-tenth) would be equivalent to four of these coins?"

KB: "Eight."

Investigator: "How did you know that?"

KB: "O.K..... alright, one-fifth is..... one-fifth would be like two-fifths compared to this (one-tenth coin) so like there'd be twice as much and it wouldn't be worth twice as much but you'd have to have twice as much to equal it to this (one-fifth coin). If you had four of these coins a fair trade would be eight of these.

Discussion

An operational system used in Task P1 to construct unit sizes and determine numbers of units entailed whole number facts, laws, and algorithms. The essential elements of the problems -- such as number of shares, the number of cookies in each share, and the total number of cookies to be shared--were related in this way. Activity involving physical experiment with material objects was an alternative solution mode. Here, a variety of equalization strategies was exhibited.

A fraction task having structural similarities to Task P1 revealed five children whose conception of the fraction 1/3 was not founded upon flexible fractional units. These children had the mental operations necessary for dealing appropriately with the problem, if fractions had been conceived of in terms of fractional units.

An operational system based on compensation logic has been advanced to explain how children coordinate relations between the sizes and numbers of units in partitions where the units consist of indefinite numbers of singletons. This system has both qualitative and quantitative dimensions. In Task P6 all children studied could tell that reducing the number of units in a partition transformation meant that the sizes of the units would be increased, and vice versa. Several children were able to say precisely by how much these units would be increased, for problems involving whole number transformation factors such as four piles to two piles, nine piles to three piles, and three piles to 12 piles. Data from corresponding fraction tasks where the underlying units also consisted of indefinite numbers of singletons shows that several children did use compensation logic to successfully solve the given problems. The children who did not succeed could not be excused by lack of intellectual mechanisms appropriate for interpreting the problems and executing solutions. They were not able to take advantage of their knowledge of units because it was not integrated into their existing fraction knowledge, or was not the basis for whatever meanings these children had for fractional number.

References

Clement, J. Cognitive microanalysis: An approach to analyzing intuitive mathematical reasoning processes. Paper presented at the Conference and Workshop on Modelling Mathematical Cognitive Development, University of Georgia, Athens, May 1979.

Hiebert, J, and Tonnessen, L. Pilot research study on fractions: Pilot Study 3. In M. Burke, N. Hiber, & T. Romberg (Eds.), The teaching of initial fraction concepts. Wisconsin R & D Center for Cognitive Learning Working Paper No. 218, Madison: The University of Wisconsin, 1977.

Hunting, R.P. The role of discrete quantity partition knowledge in the child's construction of fractional number (Doctoral dissertation, The University of Georgia, 1980) Dissertation Abstracts International, 1981(a), 41(10), 4308A-4381A (University Microfilms No 8107919).

Hunting, R.P. Qualitative compensation thought in children's solutions to fraction comparison problems. In C. Irons (Ed.) Research in Mathematics Education in Australia 1981. Volume 2. Proceedings of the Fifth Annual Meeting of the Mathematics Education Research Group of Australasia, Adelaide, May 1981(b).

Hunting, R.P. Behavioral strategies and mental processes used in solving selected partition and fraction problems. (Submitted) 1982.

Irons, C.J. An investigation of children's ability to solve six types of division problems involving sharing, sharing-implied and non-sharing situations. Volume 2 of the Proceedings of the First Conference of the Mathematics Education Research group of Australia, Monash University, Melbourne, May 1977.

Kieren, T.E. On the mathematical, cognitive, and instructional foundations of rational numbers. In R.A. Lesh (Ed.) and D.A. Bradbard (Tech. Ed.), Number and measurement: Papers from a research workshop. ERIC/SMEAC, Ohio State University, 1976.

Mitchell, C., & Blume, G. A preliminary investigation of the use of a game to identify children's use of language relevant to fractions and children's knowledge of the concept of fractions. In M. Burke, N. Hiber, & T. Romberg. The teaching of initial fraction concepts. Wisconsin R & D Center for Cognitive Learning Working Paper No. 218, Madison: The University of Wisconsin, 1977.

Piaget, J., Inhelder, B., & Szeminska, A. The child's conception of geometry. New York: Basic Books, Inc., 1960.

Schuerger, M.A., & Burke, M. Children's ability to recognize certain fractional partitionings of wholes using set and region interpretations. In M. Burke, N. Hiber, & T. Romberg (Eds.), The teaching of initial fraction concepts. Wisconsin R & D Center for Cognitive Learning Working Paper No. 218, Madison: The University of Wisconsin, 1977.